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THE LOCALIZATION OF ANGULAR MOMENTUM IN OPTICAL WAVES PROPAGATING THROUGH TURBULENCE

Darryl J. Sanchez
Denis W. Oesch

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The Localization of Angular Momentum in Optical Waves Propagating Through Turbulence

Darryl J. Sanchez[1], **Denis W. Oesch**[2]

[1] *Starfire Optical Range, AFRL/RDS, Kirtland AFB, Albuquerque, New Mexico, USA*

[2] *Science Applications International Corporation, Albuquerque, New Mexico, USA*

Abstract: This is the first in a series of papers demonstrating that photons with orbital angular momentum can be created in optical waves propagating through distributed turbulence. The scope of this first paper is much narrower. Here, we demonstrate that atmospheric turbulence imparts non-trivial angular momentum to beams and that this non-trivial angular momentum is highly localized. Furthermore, creation of this angular momentum is a normal part of propagation through atmospheric turbulence.

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1. Introduction

Recently, orbital angular momentum (OAM) has been shown [1] to occur in optical beams. That is, photons have been shown to carry a quantum effect that is traditionally associated with matter. This effect is manefested by the Poynting vector precessing hellically about the direction of propagation. Being quantum effects, they have wavefunctions and in general, the orbital angular momentum wavefunctions vary in both amplitude and phase. Specifically, the ± 1 states are given by a constant amplitude and uniform variation in azimuthal angle in phase, i.e. $e^{\pm i\phi}$ where ϕ is the azimuthal coordinate. It has been shown that the states can be created in the laboratory, but to create even the lowest order states takes careful preparation--creation of higher order states with azimuthal dependence of $e^{im\phi}$ where m is an arbitrary integer, is more difficult.

While the phase of OAM states is precisely defined, the phase in atmospheric turbulence, on the other hand, is defined by randomly varying index of refractions fluctuations. These index fluctuations impose spatially and temporally random optical path differences on optical beams passing through it. Since atmospheric density is low, these optical path differences appear as phase-only disturbances to the beam. So, beams traversing atmospheric turbulence have minimal amplitude fluctuations (propagation couples amplitude and phase, but over short distances this coupling is minimal as born out at astronomical observatories) and randomly varying phase. Being random, it appears to preclude with vanishingly small probability the helical phase patterns indicative of OAM states.

Adaptive optics is designed to compensate for these random phase distortions. Theoretical predictions of its ability to do so begin with the Helmholtz wave equation, $\nabla^2\mathbf{E} + k^2 n^2 \mathbf{E} = 0$ with $\mathbf{E}$ the electric field, k the wave number, and n the spatially varying index of refraction. The solutions to which are well known to have no angular momentum. So, it's not clear what, if any, a quantum effect would have on adaptive optics performance. However, it has been shown [2], that traveling waves can contain angular momentum and that this momentum is comprised of spin and orbital components. And in fact, the standard theoretical adaptive optic analysis [3, see for instance] initially begins with terms in the wave equation which can lead to angular momentum, but then by rough calculation, the term is discounted due to its relative size, 10^{-14} as small in the mean, vis-á-vis the other components in the wave equation. However, as is shown in Section 3 the rough calculation overlooks details which show the possibility that local effects can make the angular momentum term non-trivial.

The overall goal of this research is to demonstrate that non-trivial orbital angular momentum is created in beams propagating through atmospheric turbulence. This is doubly difficult because (1) one must show how random phase can create pure orbital angular momentum states, and (2) in measurements, the component that we would like to associate with angular momentum randomly appears and disappears from frame to frame, implying that momentum is not conserved, an impossibility. A companion paper [4] will address both of these issues and use the result of this paper to demonstrate the existence of non-trivial turbulence-induced orbital angular momentum. The scope of this paper, on the other hand, is much narrower.

The purpose of this paper is to demonstrate that using a proper wave equation, while angular momentum is negligible in the mean, because turbulence is a random field, there are locally intermittent regions where total angular momentum can be high; it is left to the companion paper to demonstrate that this is in part orbital angular momentum. To this end, Section 2 defines the terms to be used and gives an overview and summary of the approach. Section 3 demonstrates the existence of non-trivial angular momentum and the conditions required for this to be true. Finally, in Sections 4 and 5, we discuss our conclusions and areas for future work.

2. Definition of Terms and Overview/Summary of Results

The basis of our treatment is to estimate the size of the terms in the wave equation and compare the terms that create angular momentum to those that don't. To do so requires a spectral decomposition of the index fluctuations and the use of the norm of the Hilbert space of those fluctuations as the measure of size. In this way, we will show that at the highest spatial frequencies, the angular momentum becomes non-trivial.

Our physical situation is propagation though atmospheric turbulence, a weak, source-free media with the permeability of free space approximately one and a permittivity equal to the free space permittivity times the index of refraction squared. If we consider timescales long compared to the frequency of the electric field's oscillation but short compared to the evolution of atmospheric turbulence, then the electric field can be written as $\mathbf{E}(\mathbf{r})e^{i\omega t}$, that is a spatially varying part times a quickly oscillating time part. With these restrictions, the apropos wave equation is given by [3, see for instance]

$$\nabla^2 \mathbf{E} + k^2 n^2 \mathbf{E} + 2\nabla(\mathbf{E}\cdot\nabla)\log n = 0 \tag{1}$$

with k the wavenumber, n the atmospheric index of refraction. Traditionally in the field of atmospheric propagation, the last term in Equation 1 is discounted due to it's relative size. The argument goes that with the index given by $n = 1 + n_1 > 1$ with $|n_1| \leq 10^{-3}$, then at optical wavelengths $k^2 n^2 \approx 10^{12}$. Then since $\log(n) \approx n_1 \ll 1$, the third term is much less than the second term.

The problem with this argument is that the index is a random variable and the problem of taking the derivative of a random variable is ignored. Here, "size" is taken to be the norm. For a Hilbert space, the square of the norm is the inner product. In this case, the Hilbert space of index fluctuations--call it $\mathscr{H}$--is equipped with the covariance function as its inner product. (This is unusual so the point is clarified in Appendix A.) To calculate the norm, a spectral decomposition of the index fluctuations is taken and the covariance in the spectral domain calculated. Since the two spaces are isometric, this gives the desired estimate of size. This is possible since the spectrum of index fluctuations is known, and since the Fourier transform of all elements in $\mathscr{H}$ creates a conjugate Hilbert space isometric to the first [5].

Then, it is well known [2] that the angular momentum density is given by $\int \boldsymbol{\rho} \times \mathbf{E} \times \mathbf{H}\, d\boldsymbol{\rho}$ with $\boldsymbol{\rho}$ the transverse coordinates. Hence, a necessary and sufficient condition for the existence of non-zero angular momentum is

$$\hat{z}\cdot\mathbf{E} \neq 0 \tag{2}$$

with $\hat{z}$ the direction of propagation. With Equation 2 establishing the criterion, we will show in the next section that the last term in Equation 1 acts as an angular momentum source term. (Note in passing that if we had restricted ourselves to the customary Helmholtz wave equation, $\nabla^2\mathbf{E} + k^2 n^2 \mathbf{E} = 0$, then terms of the form of Equation 2 cannot occur since for any propagation given by the free space Helmholtz equation, $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to each other and perpendicular to the direction of propagation.)

As with atoms, angular momentum in photons is comprised of spin angular momentum plus orbital angular momentum. To get a lower bound of what "non-trivial" means, say the spin component is zero, and the orbital component is in the lowest non-zero state, i.e. it has $1\hbar$ of momentum. The Poynting vector precesses helically because of a component of the electric field in the $\hat{z}$ direction. It has been shown that the ratio of the $\hat{z}$ direction to the transverse direction is λ. At optical wavelengths this is approximately equal to 10^{-6}. Hence, we will look for effects of this size. We will show in the following section that because atmospheric turbulence is a random process, there exists localized regions in the atmosphere where this is false with non-trivial probability.

As the conventional argument posits, in most cases, $\nabla(\mathbf{E}\cdot\nabla)\log n$ is small. However, in the next section we show that in regions where high frequency fluctuations are present, $\nabla(\mathbf{E}\cdot\nabla)\log n$ can be non-trivially large. For any given atmosphere, the total energy in these high frequency patches is governed by the inner scale of turbulence. Calculation of the probability of finding such a patch of atmosphere is beyond the scope of this paper.

3. Algebraic Development

3.1. The Source of Angular Momentum

Beginning with a wave with no angular momentum, we show that atmospheric turbulence creates some. In what follows, this allows use of the full expression when calculating angular momentum. So, consider a plane wave incident on atmospheric turbulence. Then $\hat{z}\cdot\mathbf{E} = \hat{z}\cdot\mathbf{B} = 0$

and the third term in Equation 1 reduces in the $\hat{z}$ direction to

$$\hat{z}\cdot\nabla(\mathbf{E}\cdot\nabla)\log n(\mathbf{r}) = (\mathbf{E}\cdot\nabla)\frac{\partial \log n(\mathbf{r})}{\partial_z} \tag{3}$$

We have used the fact that the components of $\mathbf{E}$ are constant, $n(\mathbf{r})$ describes a physical process with $n(\mathbf{r}) > 1$, therefore $\log(n(\mathbf{r}))$ is smooth, so the derivatives can be interchanged. Since the index varies isotopically, the conditions for creation of angular momentum are ubiquitous. The physical cause of the appearance of this term is that as the wave interacts with the atmospheric constituents, the electric field and the displacement current are in different directions; that is, although $\nabla\cdot\mathbf{B} = 0$, here $\nabla\cdot\mathbf{E} \neq 0$, and this has the effect of scattering the componets of $\mathbf{E}$ into the $\hat{z}$ direction. Once this first interaction occurs, as shown in Equation 3, the beam contains a component in the direction of propagation and thereafter much richer interactions occur. The size of angular momentum term will be calculated under this richer condition.

3.2. *Calculation of Size, i.e. of* $||\hat{z}\cdot\nabla(\mathbf{E}\cdot\nabla)\log n||$

To calculate the size of the angular momentum term, first consider the index fluctuations. In particular, let $n(\mathbf{r}) = 1 + n_0 + n_1(\mathbf{r})$ with $n_1(\mathbf{r})$ a zero mean random variable and $n_0(\mathbf{r})$ a constant measured to be approximately 10^{-3}. Then the spectral decomposition of the varying part is [6] $n_1(\mathbf{r}) = \int d\nu(\boldsymbol{\kappa})\ e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}$ with $\int\cdot$ a Riemann-Stiejles integral. Then note, $n(\mathbf{r}) > 1$, so $|n_1(\mathbf{r})| < n_0$; hence to one part in 10^3, $\log n(\mathbf{r}) = n_0 + n_1(\mathbf{r})$. Furthermore, since $n(\mathbf{r})$ is smooth, its Fourier transform is smooth which allows the derivative and integral to be interchanged, and $\nabla \log n(\mathbf{r}) = i\int d\nu(\boldsymbol{\kappa}, z)\ \boldsymbol{\kappa}\ e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}$. Then, the norm of the source term for angular momentum is

$$||\hat{z}\cdot\nabla(\mathbf{E}\cdot\nabla\log n)||^2 = \left|\left|\left(\left[\frac{\partial}{\partial z}+1\right]\mathbf{E}\right)\cdot\left(\nabla\log n + \frac{\partial}{\partial z}\nabla\log n\right)\right|\right|^2 \tag{4}$$

We wish to study the wave as the angular momentum term builds. So, consider a small region (but big enough so that the ensemble averages are meaningful) near the turbulence boundary. In this case we use the solution to the free space wave equation as the seed beam while recognizing that it will soon contain a $\hat{z}$ term, i.e. let $\mathbf{E}(\mathbf{r}) = \mathbf{A}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}$. Then resubstituting and assuming the derivative of turbulence induced scintillation and phase is small, one obtains

$$||\hat{z}\cdot\nabla(\mathbf{E}\cdot\nabla\log n)||^2 = (1+k^2)\left[\int d\boldsymbol{\kappa}\ (1+\kappa_z^2)(\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\kappa})^2 f(\boldsymbol{\kappa})\right] \tag{5}$$

where $\boldsymbol{\kappa}$ is the three coordinates of the spatial spectrum, $k_z = 2\pi/\lambda$ since the atmosphere is isotropic, and $f(\boldsymbol{\kappa})$ the spectrum (see Appendix B). Note in passing that as the longitudinal term, $\hat{z}\cdot\mathbf{E} \neq 0$, term grows and becomes appreciable with respect to the transverse term, $\hat{z}\cdot\mathbf{E} = 0$, the assumptions here would have to be revisited and $\mathbf{E}$ be recalculated. However, this adds an additional level of complexity that does not change the main conclusion of this paper.

3.3. $||k^2n^2\mathbf{E}||$

Since $n(\mathbf{r})$ fluctuates near 1, to one part in 10^3, the size of the second term is

$$||k^2n^2\mathbf{E}||^2 = k^4A^2(\mathbf{r})\left|\left|\left(\int d\boldsymbol{\kappa}\ f(\boldsymbol{\kappa})\right)^2\right|\right| = k^4A^2(\mathbf{r}) \tag{6}$$

where the central equality shows the explicit dependence on the spectrum and the right equality the evaluation of it. Both will be used below.

3.4. *Comparison of Size – The Kolmogorov Spectrum*

Comparison of the second and third terms in Equation 1 requires a spectrum. The Kolmogorov spectrum (see Appendix B) is the most commonly used. So, substituting the Kolmogorov spectrum into Equation 5 and only considering the largest term

$$\frac{||\nabla(\mathbf{E}\cdot\nabla\log n)||^2}{||k^2n^2\mathbf{E}||^2} = \frac{k^2\int d\boldsymbol{\kappa}\ \kappa_z^2(\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\kappa})^2 f(\boldsymbol{\kappa})}{k^4A^2(\mathbf{r})\int d\boldsymbol{\kappa}\ f(\boldsymbol{\kappa})} = \frac{\lambda^2}{2\pi}\int d\boldsymbol{\kappa}\ \kappa_z^2(\hat{e}\cdot\boldsymbol{\kappa})^2 f(\boldsymbol{\kappa}) \tag{7}$$

in the $\hat{z}$ direction. In the central equality, the integral form is intentionally kept to explicitly enumerate the difference between the angular momentum and non-angular momentum terms. The most striking is the proportionality κ^4 in the $\hat{z}$ direction. This makes it strikingly apparent that the third term in Equation 1 is dominated by high spatial frequencies. In the right most equality, the aforementioned proportionality to λ^2--for which this ratio is typically discounted--appears. Since the inertial range of the Kolmogorov spectrum is unbounded, i.e. the spectrum is $-11/3$ to all scales, there exists a frequency in the spectrum above which the numerator dominates the denominator, i.e. for which the angular momentum term dominates the customary term. In particular,

$$\frac{||\nabla(\mathbf{E}\cdot\nabla\log n)||^2}{||k^2n^2\mathbf{E}||^2} \to \infty \quad \text{for } \kappa \text{ large} \tag{8}$$

That is, the size of the term that produces angular momentum can become arbitrarily larger than the customary term that is kept.

However, to do so requires arbitrarily high spatial frequencies, and such structures are not supported in real atmospheres. So, let's consider a more physical situation. (But also note that as the angular momentum term grows, at some point it is comparable in size to the customary terms and the assuptions leading to Equation 5 would have to be revisited.)

3.5. *Comparison of Size – The von Karman Spectrum*

The von Karman spectrum is another commonly used spectrum (see Appendix B). This spectrum limits both the upper and lower frequencies. For our purposes here, it exponentially extinguishes frequencies higher than a characteristic frequency, call it κ_i [3]. Substituting into Equation 7

$$\frac{||\nabla(\mathbf{E}\cdot\nabla\log n)||^2}{||k^2n^2\mathbf{E}||^2} \propto \lambda^2\int d\kappa\ \kappa^4\ (\kappa^2+\kappa_o^2)^{-11/6}\exp\left(-\frac{\kappa^2}{\kappa_i^2}\right) \tag{9}$$

Immediately obvious, unlike the Komogorov spectrum, this ratio is finite in all cases, but also note, there is an amplification at the frequencies closest to κ_i. Since for the atmosphere, $\kappa_i \gg 1$, the value of the numerator will be dominated by frequencies closest to κ_i. Whether it is non-trivial is solely a function of κ_i.

κ_i is the inverse of the inner scale of turbulence, L_i. The inner scale is the characteristic length at which the atmosphere becomes viscous and turbulence (motion) gets converted to heat. Measurements have found L_i to be $1-5mm$ at sea level and centimeters at higher altitudes. Following standard practice, consider κ_i constant; then the integrals can be performed. This is done numerically for three values of the innerscale, $L_i = \{0.1mm, 1mm, 10mm\}$; these values bound the range of typically measured inner scales. The values of Equation 9 are $\{55, 2.7, 0.12\}\times 10^{-6}$ respectively and recall that 10^{-6} would be significant. Based on this rough calculation, the effect is on the cusp of detectability.

But recall, this analysis is performed near the turbulence layer boundary with a patch size large enough to give reasonable statistics. Since the atmospheric outer scale--the smallest patch giving proper statistics at all frequencies--for astronomical imaging is approximately $10m$, consider patches of this size. Then propagation over $1Km$ will yield 100 independent source patches of angular momentum, each contributing equally as a source. In this example, the value of the ratio in Equation9 is multiplied 100-fold. And furthermore, since propagation through turbulence is not restricted to $1Km$, the ratio could grow very much bigger.

Hence, we have shown that there is a viable mechanism for creation of non-trivial angular momentum in beams and that this will occur in the localized regions of high spatial frequencies.

4. Discussion

The purpose of this paper is to demonstrate that beginning with the full wave equation, while angular momentum is negligable in the mean, that because turbulence is a random field, there can be local regions where it is high. Equations 7 and 9 demonstrate this at the highest spatial frequencies.

4.1. The Probability of non-zero Angular Momentum

An immediate question then is "what is the probability for a given atmosphere that the inner scale will be of that size?". So, while we have shown that non-trivial angular momentum can occur, calculating the probability of such as occurrence is another matter. Calculating the probability is equivalent to calculating the probability that $\kappa > \kappa^{\text{thresh}}$ given that $\kappa_i < \kappa^{\text{thresh}}$ where κ^{thresh} is the threshold for detection of angular momentum. This calculation is the complementary problem to the Fried "Lucky Imaging Problem" [7]. Whereas there, Fried calculated the probability that locally (over the telescope diameter) the atmosphere is unusually quiescent, i.e. is comprised mostly of low frequency components, here, we are interested in calculating the probability that locally (over the atmosphere interrogated by an optical beam) the atmosphere is unusually active. Enumerating this calculation is beyond the scope of this paper.

4.2. The Innerscale and κ_i

The inner scale is the characteristic length at which the atmosphere becomes viscous and wind velocity is converted into heat. We have interest in this only to the extent that this mechanism quashes the high frequency fluctuations responsible for the creation of angular momentum. Since based on our expressions, the inner scale dominates at the very frequencies where the effect occurs, factors of two can be important to this effect occurring on measurable scales or not. Unfortunately, there are no known references for the physics of how the highest spatial frequency components are dissipated. So, while here we follow conventional notation and assign a frequency to the inverse characteristic scale of heat dissipation, on physical grounds, this must be a gross simplification. How, for instance, do the highest frequencies form and how does the viscousity dissipate them; both of these are temporal effects, yet this is not considered here. For the purposes of this paper, this is merely noted as an area of further research.

4.3. Further Work

Much work remains to be done. Some of this is presented in a companion paper, where it is demonstrated that the localized angular momentum is in part orbital angular momentum, and that this orbital angular momentum is equivalent to the measurement of branch points. So, even though the source term in Equation 1 may lead to a (perhaps small) E_z term locally, propagation through extended turbulence will cause this term to accumulate. In fact, it is well known that propagation through extended turbulence causes branch points to appear, and that propagation through very much extended turbulence causes the branch point density to grow to the extend that they are a nuisance to adaptive optic performance--a nice corrobaration of results and insight into the physical process of formation of angular momentum.

Some remaining work is still open. For instance, calculation of the probability of $\kappa > \kappa^{\text{thresh}}$ is beyond the scope of this paper. Also, the result here applies to a local patch; once the longitudinal component is non-trivial with respect to the transverse components, the expressions leading to Equation 4 must be reevaluated; this is beyond the scope of this paper.

5. Summary

We have demonstrated that atmospheric turbulence imparts non-trivial angular momentum to beams and that this is a normal part of progagation through turbulence. This non-trivial angular momentum is highly localized since it is caused by the highest fluctating regimes in the turbulence.

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A. The Hilbert Space of Index Fluctuations

Atmospheric turbulence is described by a spatially varying perimittivity, $\varepsilon(\mathbf{r})$. It is related to the index of refraction as $\varepsilon(\mathbf{r}) = \varepsilon_0(\mathbf{r})n^2(\mathbf{r})$ with ε_0 the permitivitty of free space and n the index of refraction. For the atmosphere, the index can be written (for clarity dropping the spatial dependance),

$$n = 1 + n_0 + n_1$$

with $n > 1$, n_0 a constant $\approx 10^{-3}$, and n_1 a varying term with a zero mean. Because this index describes atmospheric fluctuations, n is real, $n > 1$, and $E(n_1) = 0$ with $E(\cdot)$ the expectation operator.

A.1. Construction of the Hilbert Space

Consider the set of all possible index fluctuations, $\{n_1\}$. Since n_1 describes a physical process, $n_1 \in L_2$ with L_2 the space of square integrable functions. Note, that since we are describing atmospheric fluctuations which by definition does not include the vacuum, $0 \notin \{n_1\}$. Also note, that $\{n_1\} \subset L_2$ since the atmosphere is finite, $max(\{n_1\}) \ll 1$, and $min(\{n_1\}) > -n_0$.

Complete the set $\{n_1\}$. Since L_2 is complete and since $\{n_1\} \subset L_2$, this completion necessarily lies in L_2. Choose the smallest such completion and call it $\mathscr{H}_n$. $\mathscr{H}_n$ is a Hilbert space.

A.2. The Covariance as an Inner Product

Let the notation $(\cdot,\cdot)$ denote the inner product. Then, the definition of the inner product is (Reed and Simon, p. 36)

$$\begin{array}{rll} (i) & (x,x) \geq 0 & \text{and } (x,x) = 0 \text{ iff } x = 0 \\ (ii) & (x,y+z) = (x,x) + (x,z) & \\ (iii) & (x,\alpha y) = \alpha(x,y) & \\ (iv) & (x,y) = (y,x)^* & \end{array} \tag{10}$$

where V is a vector space, $x,y,z \in V$, and $\alpha \in \mathbb{C}$

Let $R(\cdot,\cdot)$ denote the covariance function. Then, the covariance function is defined as (Gikhman, p. 9)

$$R(x,y) = M([\xi(y) - M(\xi(y))][\xi(x) - M(\xi(x))]^*)$$

with $R(x,y)$ the covariance function, $M(\cdot)$ the expectation operator, * the complex conjugate, and $\xi(\cdot)$ a function that maps $V \to V$. For our purposes, let $\xi(x) = x$. Then,

$$R(x,y) = M([y - M(y)][x - M(x)]^*).$$

Gikhman shows

$$\begin{array}{rll}
(i) & R(x,x) \geq 0 & \text{and } R(x,x) = 0 \text{ iff } \xi(x) = \text{constant} \\
(ii) & R(x,y) = R^*(y,x) & \\
(iii) & |R(x,y)|^2 = R(x,x)R(y,y) & \\
(ii) & \forall n, x_1, \ldots, x_n \in V \text{ and } \lambda_1, \ldots \lambda_n \in \mathbb{C}, \displaystyle\sum_{j,k=1}^{n} R(x_j,x_k)\lambda_j\lambda_k^* \geq 0 &
\end{array}$$

To show that $R(\cdot,\cdot)$ is an inner product on $\mathscr{H}_n$, we will show the four parts of the definition in Equation 10. To begin, let $x,y,z \in \mathscr{H}_n$. So, $M(x) = M(y) = M(z) = 0$. Using the definition of the covariance function, it is trivial to show the first part of (i), and also (ii), (iii), and (iv) in the definition, namely

$$\begin{aligned}
R(x,x) &= M(x^2) > 0 \\
R(x,y+z) &= R(x,y) + R(x,z) \\
R(x,ay) &= aR(x,y) \\
R(x,y) &= (R(y,x))^*
\end{aligned}$$

The only point needing clarification is the second part of (i), i.e.

$$R(x,x) = 0 \quad \Leftrightarrow \quad x = 0.$$

Since we are describing atmospheric fluctuations, $0 \notin \{n_1\}$. But note, $\mathscr{H}_n$ is the completion of $\{n_1\}$ and $0 \in \mathscr{H}_n$. Then since $M(x^2) > 0$, $R(x,x) = 0$ if and only if $x = 0$.

Following Ghikman, $\mathscr{H}_n$ is a Hilbert space of a random process (the atmospheric fluctuations). The Fourier transform of this space is also a Hilbert space and is isometric to the first. So, the norm (inner product) in $\mathscr{H}_n$ equals the norm (inner product) of the transformed variable in the other. In the text, the norm is always evaluated in the transformed space.

B. The Spectrum of Turbulence

Based on Kolmogorov's dimensional analysis, it can be shown that the power spectrum in the inertial range is given by

$$f(\kappa) = C\kappa^{-11/3} \tag{11}$$

where κ is the spatial frequency. This is the Kolmogorov spectrum.

Physically, spatially there are upper bounds, L_o, and lower bounds, L_i, that the atmosphere can attain. This is captured by the von Karman spectrum

$$f(\kappa) = C(\kappa^2 + \kappa_o^2)^{-11/6} \exp\left(-\frac{\kappa^2}{\kappa_i^2}\right) \tag{12}$$

with $\kappa_i = \frac{2\pi}{L_i}$ the inner scale (in frequency) of turbulence, $\kappa_o = \frac{2\pi}{L_o}$ the outer scale (in frequency), and L_i and L_o are respectively the inner and outer scale. The inner scale is the length at which turbulence gets dissapated by heat. For our purposes, the inner scale is the smallest scale at which turbulence structures are supported.

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